Quantumphysics 2

Exam August 23, 2010.

Tentamenhal 02 (blauwborgje 4), 9.00-12.00.

◇ *Write your name and student number on* <u>*each*</u> *sheet you use.*
◇ *The exam has 4 problems.*
◇ *Read the problems carefully and give complete and readable answers.*
◇ *No books or personal notes are allowed.*

Problem 1

i) What are the cartesian-coordinate representations of the angular momentum operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$? Using this, prove the commutation relation $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$.

[4 points]

Answer: We have that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, hence

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),$$

and the same for the other components, cyclically permuting x, y, z.

ii) Show that the operators $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ act respectively as raising and lowering operators for the z component of the angular momentum, by evaluating the action of $\hat{L}_z$ on the states $\hat{L}_\pm|l,m\rangle$. *(hint: the commutator* $[L_\pm, L_z]$ *might be useful here)*

[4 points]

Answer: We need to consider $L_zL_\pm$, that is $L_\pm L_z - [L_\pm, L_z]$. The required commutator is $[L_\pm, L_z] = \mp\hbar L_\pm$. Thus $L_zL_\pm|l,m\rangle = (L_\pm L_z \pm \hbar L_\pm)|l,m\rangle$. Since $|l,m\rangle$ is an eigenstate of L_z, this gives $L_zL_\pm|l,m\rangle = \hbar(m \pm 1)L_\pm|l,m\rangle$. Therefore $L_\pm|l,m\rangle$ is eigenstate for the operator L_z but with eigenvalues $\hbar(m \pm 1)$, i.e. $L_\pm$ is a raising/lowering operator for L_z.

iii) A system is in an eigenstate of $\hat{L}^2$ and $\hat{L}_z$, with quantum numbers l and m. Calculate $\langle\hat{L}_x\rangle$ and $\langle\hat{L}_x^2\rangle$.

[3 points]

Answer: Simply applying $L_x = \frac{L_+ + L_-}{2}$ to the ψ state, we find out that $\langle L_x\rangle = 0$. Writing $L_x^2 = (L_+^2 + L_-^2 + L_+L_- + L_-L_+)/4$ and again applying the raising/lowering properties, we find out that $\langle L_x^2\rangle = \hbar^2[l(l+1) - m^2]/2$.

iv) The wavefunction of a hydrogen atom can be written as $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi)$. Sketch the radial parts $R_{00}(r)$, $R_{20}(r)$, and $R_{21}(r)$.

[3 points]

Answer: R_{00} is eindig voor $r = 0$ en heeft geen nuldoorgangen; R_{10} is eindig voor $r = 0$ en heeft een nuldoorgang; R_{10} is 0 voor $r = 0$ en heeft geen nuldoorgangen

v) Give the matrix representation for the operators $\hat{S}_z$ and $\hat{S}^2$ for a particle with spin $3/2$.

[3 points]

Answer:

$$S^2 = \frac{15\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

vi) An unperturbed system has only two eigenfunctions $|1\rangle = xe^{-x^2}$ and $|2\rangle = x^3e^{-x^2}$ which are non-degenerate. Suppose there is a perturbing potential of the form αx. Argue (*i.e. do not calculate!*) why this perturbation does not lead to a first or second order correction to either the energy or the wavefunctions.

[3 points]

Answer: All the integrals involved in the perturbation calculations are vanishing because we are integrating an odd function on a even domain. So the perturbation does not change the energetic levels.

vii) Why can the matrix below not be a valid density matrix?

[3 points]

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Answer: The trace is not equal to 1

Problem 2

Consider a tritium atom, consisting of a nucleus ^{3}H (the triton) an an electron. The triton, which consists of a proton ($Z = 1$) and two neutrons, is very unstable, since by β-emission it decays to ^{3}He, which contains two protons ($Z = 2$) and a neutron. This decay process occurs very rapidly with respect to the characteristic atomic times and can be considered instantaneous. As a result, there is a sudden doubling of the Coulomb attraction between the atomic electron and the nucleus, when the decay takes place.

Assuming that the tritium atom is in the ground state $\psi_{1s}^{(Z=1)}(\vec{r})$ when the decay takes place, and neglecting recoil effect, find the probability that immediately after the decay the He$^+$ ion can be found:

i) In its ground state $\psi_{1s}^{(Z=2)}(\vec{r})$.

[7 points]

Answer: Let's assume that the decay takes place at $t = 0$. At times $t < 0$, the wavefunction of the system is

$$\Psi(\mathbf{r}, t) = \psi_{1s}^{Z=1}(r)\exp(-iE_{1s}t),$$

where

$$\psi_{1s}^{Z=1}(r) = \frac{1}{\sqrt{\pi}}\exp(-r)$$

is the ground state for the tritium atom. At times $t > 0$, we have

$$\Psi(\mathbf{r}, t) = \sum_k c_k \psi_k^{Z=2}(\mathbf{r})\exp(-iE_k t),$$

where $\psi_k^{Z=2}$ are the hydrogenic wavefunctions for $Z = 2$ (^{3}He). Thus the probability of finding the ^{3}He in the $\psi_k^{Z=2}$ at $t > 0$ id $P_k = |c_k|^2$, where $c_k = \langle\psi_k^{Z=2}|\psi_{1s}^{Z=1}\rangle$. In the specific case,

$$c_{1s} = \int [\psi_{1s}^{Z=2}(r)]^* \psi_{1s}^{Z=1}(r)\, d\mathbf{r} = \frac{16\sqrt{2}}{27}$$

. Therefore the probability is $P_{1s} = |c_{1s}|^2 = 0.702$

ii) In any state other than the ground state (i.e. total probability for excitation or ionization).

[3 points]

Answer: The total probability for excitation or ionization is $1 - P_{1s} = 0.298$

iii) In the 2s state.

[5 points]

Answer: $P_{2s} = 0.25$

iv) In a state with $l \neq 0$

[5 points]

Answer: For states with $l \neq 0$, the coefficients c_k vanish, since $\psi_{1s}^{Z=1}$ is spherically symmetric. Thus $P_k = 0$

Hint: The s wavefunctions for $n = 1, 2$ are given by:

$$\psi_{1s}(\vec{r}) = \frac{1}{\sqrt{\pi}}(Z/a_0)^{3/2}\exp(-Zr/a_0)$$

$$\psi_{2s}(\vec{r}) = \frac{1}{2\sqrt{2\pi}}(Z/a_0)^{3/2}(1 - Zr/2a_0)\exp(-Zr/2a_0).$$

The integral

$$\int_0^\infty dr\, r^m e^{-xr} = m! x^{-(m+1)}$$

might be useful.

Problem 3

A one-dimensional anharmonic oscillator is described by the Hamilton operator

$$\hat{H} = \hat{H}_0 + Ax^3 + Bx^4,$$

where

$$\hat{H}_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

i) Give the general expression for the first order energy correctie of the $n-$th energy level of the unperturbed system.

[4 points]

Answer: De eerste orde correctie voor het n-de niveau zijn gelijk aan de diagonaal componenten van $\hat{H}'$: $\left\langle n \middle| \hat{H}' \middle| n \right\rangle$.

ii) Calculate the first order correction to the unperturbed $n-$th energy level due to the presence of the perturbation.

[8 points]

Answer: De operator $\hat{x}$ kan uitgedrukt worden in termen van creatie en annihilatie operatoren: $\hat{x} = \frac{1}{\beta\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right)$, where $\beta^2 = \frac{m\omega_0}{\hbar}$. Hiervan gebruikmakend wordt de uitdrukking voor $\hat{x}^3$:

$$\hat{x}^3 = \left(\frac{1}{\beta\sqrt{2}}\right)^3 \left[\left(\hat{a}^\dagger\right)^3 + \hat{a}^\dagger\hat{a}^2 + \left(\hat{a}^\dagger\right)^2\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\left(\hat{a}^\dagger\right)^2 + \hat{a}^3 + \hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^2\hat{a}^\dagger\right]$$

Voor de eerste orde correcties moeten we de diagonaal elementen uitrekenen:

$$\left\langle n|\hat{H}'|n\right\rangle = A\left\langle n|\hat{x}^3|n\right\rangle + B\left\langle n|\hat{x}^4|n\right\rangle.$$

Omdat de kubische term altijd een oneven aantal creatie/annihilatie operatoren bevat zijn de bijdrages aan de diagonaal elementen hiervan gelijk aan nul.
Voor de andere bijdrage (x^4 term) geldt:

$$\begin{aligned}\left\langle n|\hat{x}^4|n\right\rangle = \left(\frac{1}{\beta\sqrt{2}}\right)^4 \Big[\left\langle n\left|\left(\hat{a}^\dagger\right)^2\hat{a}^2\right|n\right\rangle + \left\langle n\left|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\right|n\right\rangle + \left\langle n\left|\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\right|\right\rangle +& \\ \left\langle n\left|\hat{a}\left(\hat{a}^\dagger\right)^2\hat{a}\right|n\right\rangle + \left\langle n\left|\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\right|n\right\rangle + \left\langle n\left|\hat{a}^2\left(\hat{a}^\dagger\right)^2\right|n\right\rangle\Big] =& \\ \left(\frac{1}{\beta\sqrt{2}}\right)^4 \Big[\sqrt{n(n-1)(n-1)n} + \sqrt{nnnn} + \sqrt{(n+1)(n+1)nn} +& \\ \sqrt{nn(n+1)(n+1)} + \sqrt{(n+1)(n+1)(n+1)(n+1)} +& \\ \sqrt{(n+1)(n+2)(n+2)(n+1)}\Big] =& \\ \left(\frac{1}{\beta\sqrt{2}}\right)^4 (6n^2+6n+3) = \frac{3}{4\beta^4}(2n^2+2n+1)&\end{aligned}$$

Dus de energie correctie voor het n-de niveau is $\frac{3B}{4\beta^4}(2n^2+2n+1)$.

iii) Under what conditions are the corrections found in ii) a good approximation? (this will depend on the quantum number n.)

[8 points]

Answer: Storingsrekening is een goede benadering als geldt:

$$\left|\hat{H}'_{nn}\right| << E_n^0$$

en

$$\left|\hat{H}'_{nn}\right| << \left|E_n^0 - E_{n\pm1}^0\right|.$$

De niveauopsplitsing tussen opeenvolgende niveaus voor de harmonische oscillator is $E_n^0 - E_{n-1}^0 = \hbar\omega$. Verder geldt $\left|\hat{H}'_{nn}\right| \approx \frac{3B}{2\beta^4}n^2$, dus

$$\frac{3B}{2\beta^4}n^2 << \hbar\omega.$$

Hint: $\hat{x} = \frac{1}{\beta\sqrt{2}}(\hat{a}^\dagger + \hat{a})$

Problem 4

Consider the two state system of a spin 1/2 particle, where the eigenstates of S_z are given by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Suppose there is a magnetic field $\vec{B}$ pointing in the z direction, $\vec{B} = (0, 0, B)$, and a corresponding Hamiltonian given by $\hat{H} = -\vec{B} \cdot \vec{\mu}$, where $\vec{\mu} = -\frac{e}{mc}\vec{S}$ and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

i) Find the normalized energy eigenstates and eigenvalues.

[5 points]

Answer: The Hamiltonian is $\hat{H} = \frac{eB}{2mc}\sigma_z$, so clearly $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\hat{H}$. Since $\sigma_z|\uparrow\rangle = +|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$, we have that

$$\hat{H}|\uparrow\rangle = +\frac{eB}{2mc}|\uparrow\rangle \qquad \hat{H}|\downarrow\rangle = +\frac{eB}{2mc}|\downarrow\rangle$$

ii) Find the normalized eigenstates and eigenvalues of $\hat{S}_x$ in terms of the eigenstates of $\hat{S}_z$.

[6 points]

Answer: The normalized states for $S_x = \frac{\hbar}{2}\sigma_x$ are

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

with $S_x|+\rangle = +\frac{\hbar}{2}|+\rangle$ and $S_x|-\rangle = -\frac{\hbar}{2}|-\rangle$

Assume that at time $t = 0$ the spin state is the positive eigenstate of $\hat{S}_x$.

iii) Find the state as a function of time, t.

[5 points]

Answer: Let $|\chi, t\rangle$ be the time dependent state, so we have

$$|\chi, 0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

At later times the state is

$$|\chi, t\rangle = e^{-iHt/\hbar}|\chi, 0\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle\right)$$

with $\omega = \frac{eB}{mc}$.

iv) Find the expectation value $\langle \hat{S}_x \rangle$ as a function of t for the state in iii).

[6 points]

Answer: Using $\sigma_x|\uparrow\rangle = |\downarrow\rangle$ and $\sigma_x|\downarrow\rangle = |\uparrow\rangle$ one finds:

$$\begin{aligned}
\langle \chi, t|S_x|\chi, t\rangle &= \frac{\hbar}{2}\frac{1}{2}\left(\langle\uparrow|e^{i\omega t/2} + \langle\downarrow|e^{-i\omega t/2}\right)\sigma_x\left(e^{-i\omega t/2}|\uparrow\rangle + e^{i\omega t/2}|\downarrow\rangle\right) \\
&= \frac{\hbar}{4}\left(\langle\uparrow|e^{i\omega t/2} + \langle\downarrow|e^{-i\omega t/2}\right)\left(e^{-i\omega t/2}|\downarrow\rangle + e^{i\omega t/2}|\uparrow\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{i\omega t} + e^{-i\omega t}\right) = \frac{\hbar}{2}\cos\omega t
\end{aligned}$$

v) Find the probabilitity as a function of t that a measurement of S_x will give a positive outcome for the state found in iii).

[5 points]

Answer: Using

$$| \uparrow\rangle = (|+\rangle + |-\rangle) \qquad\qquad | \downarrow\rangle = (|+\rangle - |-\rangle)$$

we have

$$\begin{aligned} |\chi, t\rangle &= \frac{1}{2}\left(\left(e^{-i\omega t/2} + e^{i\omega t/2}\right)|+\rangle + \left(e^{-i\omega t/2} - e^{i\omega t/2}\right)|-\rangle\right) \\ &= \cos\frac{1}{2}\omega t|+\rangle - i\sin\frac{1}{2}\omega t|-\rangle. \end{aligned}$$

So the probability to find a positive outcome is $\cos^2 \frac{1}{2}\omega t$